

ON THE APPROXIMATION BY CONVOLUTION TYPE DOUBLE SINGULAR INTEGRAL OPERATORS

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ABSTRACT. In this paper, we prove the pointwise convergence and the rate of pointwise convergence for a family of singular integral operators in two-dimensional setting in the following form:

$$L_{\lambda}(f; x, y) = \iint_D f(t, s) K_{\lambda}(t - x, s - y) ds dt, \quad (x, y) \in D,$$

where $D = \langle a, b \rangle \times \langle c, d \rangle$ is an arbitrary closed, semi-closed or open rectangle in \mathbb{R}^2 and $\lambda \in \Lambda$, Λ is a set of non-negative numbers with accumulation point λ_0 . Also, we provide an example to support these theoretical results. In contrast to previous works, the kernel function $K_{\lambda}(t, s)$ does not have to be even, positive or 2π -periodic.

1. Introduction

Taberski [7] studied the pointwise convergence of integrable functions and the approximation properties of derivatives of integrable functions in $L_1(-\pi, \pi)$ by a family of convolution type singular integral operators depending on two parameters in the following form:

$$(1.1) \quad L_{\lambda}(f; x) = \int_{-\pi}^{\pi} f(t) K_{\lambda}(t - x) dt, \quad x \in (-\pi, \pi), \quad \lambda \in \Lambda,$$

where $K_{\lambda}(t)$ is the kernel satisfying suitable assumptions and Λ is a given set of non-negative numbers with accumulation point λ_0 .

After Taberski's study [7], Gadjiev [3] and Rydzewska [9] gave some approximation theorems concerning pointwise convergence and the order of pointwise convergence for operators of type (1.1) at a generalized Lebesgue point and a μ -generalized Lebesgue point of $f \in L_1(-\pi, \pi)$, respectively. Later on, Bardaro [1] estimated the degree of pointwise convergence of general type singular integrals at generalized Lebesgue points of the functions $f \in L_1(\mathbb{R})$.

In [8], Taberski explored the pointwise convergence of integrable functions in $L_1(Q)$ by a three-parameter family of convolution type singular integral operators of the form:

$$(1.2) \quad L_{\lambda}(f; x, y) = \iint_Q f(t, s) K_{\lambda}(t - x, s - y) ds dt, \quad (x, y) \in Q,$$

1991 *Mathematics Subject Classification.* Primary 41A35; Secondary 41A25.

Key words and phrases. generalized Lebesgue point; pointwise convergence; rate of convergence.

where $Q = \langle -\pi, \pi \rangle \times \langle -\pi, \pi \rangle$ is a closed, semi-closed or open rectangle. In this work, the kernel function K_λ was non-negative, even and 2π -periodic with respect to both variables, separately. Then, Rydzewska [10] extended this work by obtaining the rate of pointwise convergence at a μ -generalized d -point. Here, note that all other assumptions on the indicated operators were same with [8]. In particular, for further studies on the convergence of double singular integral operators, we address the reader to [11, 12, 13, 4].

In this study, we also investigated the pointwise convergence and the rate of convergence of the operators similar to the studies above. In contrast to previous works, the kernel function $K_\lambda(t, s)$ does not have to be even, non-negative and 2π -periodic with respect to each variable.

The main contribution of this paper is to investigate the rate of pointwise convergence of the convolution type singular integral operators in the following form:

$$(1.3) \quad L_\lambda(f; x, y) = \iint_D f(t, s) K_\lambda(t - x, s - y) ds dt, \quad (x, y) \in D$$

where $D = \langle a, b \rangle \times \langle c, d \rangle$ is an arbitrary closed, semi-closed or open bounded rectangle in \mathbb{R}^2 , to $f \in L_1(D)$, at a μ -generalized Lebesgue point as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$. Here, $L_1(D)$ is the collection of all measurable functions f for which $|f|$ is integrable on D , $\Lambda \subset \bar{\mathbb{R}}$ is a set of non-negative indices with accumulation point λ_0 .

The paper is organized as follows: In Section 2, we introduce the fundamental definitions. In Section 3, we prove the existence of the operators of type (1.3). In Section 4, we present two theorems concerning the pointwise convergence of $L_\lambda(f; x, y)$ to $f(x_0, y_0)$ whenever (x_0, y_0) is a μ -generalized Lebesgue point of f , for the cases D is bounded rectangle and $D = \mathbb{R}^2$. In Section 5, we establish the rate of convergence of operators of type (1.3) to $f(x_0, y_0)$ as (x, y, λ) tends to (x_0, y_0, λ_0) and we conclude the paper with an example to support our results.

2. PRELIMINARIES

In this section we introduce the main definitions used in this paper.

The following definition is obtained by modifying the μ -generalized Lebesgue point definition given in [10].

Definition 1. A point $(x_0, y_0) \in D$ is called a μ -generalized Lebesgue point of function $f \in L_1(D)$ if

$$\lim_{(h,k) \rightarrow (0,0)} \frac{1}{\mu_1(h)\mu_2(k)} \int_0^h \int_0^k |f(t + x_0, s + y_0) - f(x_0, y_0)| ds dt = 0,$$

where $\mu_1(h) = \int_0^h \rho_1(t) dt > 0$, $0 < h < \delta_0 < \min\{b - a, d - c\}$ and $\rho_1(t)$ is an

integrable and non-negative function on $[0, \delta_0]$ and similarly, $\mu_2(k) = \int_0^k \rho_2(s) ds > 0$, $0 < k < \delta_0 < \min\{b - a, d - c\}$ and $\rho_2(s)$ is an integrable and non-negative function on $[0, \delta_0]$.

Now, we define the new set of kernel functions by using some of the kernel conditions presented in [11].

Definition 2. (Class A) We will say that the function $K_\lambda(t, s)$ belongs to class A, if the following criterions are fulfilled:

- a: $K_\lambda(t, s)$ is defined and integrable as a function of (t, s) on \mathbb{R}^2 for each fixed $\lambda \in \Lambda$ and $\|K_\lambda\|_{L_1(\mathbb{R}^2)} \leq M, \forall \lambda \in \Lambda$.
- b: For fixed $(t_0, s_0) \in D$, $K_\lambda(t_0, s_0)$ tends to infinity as λ tends to λ_0 .
- c: $\lim_{(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)} \left| \iint_{\mathbb{R}^2} K_\lambda(t - x, s - y) ds dt - 1 \right| = 0$.
- d: $\lim_{\lambda \rightarrow \lambda_0} |K_\lambda(\gamma, 0)| = \lim_{\lambda \rightarrow \lambda_0} |K_\lambda(0, \gamma)| = 0, \forall \gamma > 0$.
- e: $\lim_{\lambda \rightarrow \lambda_0} \iint_{\mathbb{R}^2 \setminus \langle -\gamma, \gamma \rangle \times \langle -\gamma, \gamma \rangle} |K_\lambda(t, s)| ds dt = 0, \forall \gamma > 0$.
- f: There exist the numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|K_\lambda(t, s)|$ is monotonically increasing with respect to t on $(-\delta_1, 0]$ and monotonically decreasing on $[0, \delta_1)$ and similarly $|K_\lambda(t, s)|$ is monotonically increasing with respect to s on $(-\delta_2, 0]$ and monotonically decreasing on $[0, \delta_2)$ for any $\lambda \in \Lambda$. Similarly, $|K_\lambda(t, s)|$ is bimonotonically increasing with respect to (t, s) on $[0, \delta_1) \times [0, \delta_2)$ and $(-\delta_1, 0] \times (-\delta_2, 0]$ and bimonotonically decreasing with respect to (t, s) on $[0, \delta_1) \times (\delta_2, 0]$ and $(-\delta_1, 0] \times [0, \delta_2)$ for any $\lambda \in \Lambda$.

Remark 1. If the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is bimonotonic on $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \subset \mathbb{R}^2$, then the equality given by

$$\begin{aligned} V(g; [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]) &= \bigvee_{\alpha_1}^{\alpha_2} \bigvee_{\beta_1}^{\beta_2} (g(t, s)) \\ &= |g(\alpha_1, \beta_1) - g(\alpha_1, \beta_2) - g(\alpha_2, \beta_1) + g(\alpha_2, \beta_2)| \end{aligned}$$

holds [8, 5].

Throughout this paper, we suppose that the kernel $K_\lambda(t, s)$ belongs to class A and $\delta_0 \leq \min\{\delta_1, \delta_2\}$.

3. Existence of the operators

Lemma 1. If $f \in L_1(D)$, then the operator $L_\lambda(f; x, y)$ defines a continuous transformation acting on $L_1(D)$.

Proof. Since $L_\lambda(f; x, y)$ is linear, it is sufficient to show that the expression given by

$$\|L_\lambda\|_1 = \sup_{f \neq 0} \frac{\|L_\lambda(f; x, y)\|_{L_1(D)}}{\|f\|_{L_1(D)}}$$

is bounded.

Let $D = \langle a, b \rangle \times \langle c, d \rangle$ be an arbitrary closed, semi-closed or open bounded rectangle in \mathbb{R}^2 . The function f is extended to \mathbb{R}^2 by defining g such that

$$g(t, s) = \begin{cases} f(t, s), & \text{if } (t, s) \in D, \\ 0, & \text{if } (t, s) \in \mathbb{R}^2 \setminus D. \end{cases}$$

Now, using Fubini's Theorem [2], we have

$$\begin{aligned}
\|L_\lambda(f; x, y)\|_{L_1(D)} &= \iint_D \left| \iint_{\mathbb{R}^2} g(t, s) K_\lambda(t - x, s - y) ds dt \right| dy dx \\
&\leq \iint_D \left(\iint_{\mathbb{R}^2} |g(t + x, s + y)| |K_\lambda(t - x, s - y)| ds dt \right) dy dx \\
&= \iint_{\mathbb{R}^2} |K_\lambda(t - x, s - y)| \left(\iint_D |g(t + x, s + y)| dx dy \right) dy dx \\
&= \iint_{\mathbb{R}^2} |K_\lambda(t - x, s - y)| \left(\iint_{\mathbb{R}^2} |g(t + x, s + y)| dx dy \right) dy dx \\
&\leq M \|f\|_{L_1(D)}.
\end{aligned}$$

One may prove the assertion for the case $D = \mathbb{R}^2$ using similar method. The proof is completed. \square

4. Pointwise convergence

The following theorem gives a pointwise approximation of the integral operators of type (1.3) to the function f at μ -generalized Lebesgue point of $f \in L_1(D)$ whenever D is an arbitrary rectangle in \mathbb{R}^2 such that bounded, closed, semi-closed or open.

Theorem 1. *If (x_0, y_0) is a μ -generalized Lebesgue point of function $f \in L_1(D)$, then*

$$\lim_{(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)} L_\lambda(f; x, y) = f(x_0, y_0)$$

on any set Z on which the functions

$$(4.1) \quad \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} \int |K_\lambda(t - x, s - y)| \rho_1(|t - x_0|) \rho_2(|s - y_0|) ds dt,$$

and

$$(4.2) \quad |K_\lambda(0, 0)| \mu_1(|x - x_0|) \quad \text{and} \quad |K_\lambda(0, 0)| \mu_2(|y - y_0|)$$

are bounded as (x, y, λ) tends to (x_0, y_0, λ_0) . Here, the set Z consists of the points $(x, y, \lambda) \in D \times \Lambda$ such that the functions given by (4.1) and (4.2) are bounded for all $\delta > 0$ which satisfy $0 < \delta < \delta_0$.

Proof. The proof begins with the following compulsory assumptions: Suppose that $(x_0, y_0) \in D$, $0 < x_0 - x < \delta/2$, for all $\delta > 0$ which satisfies $x_0 + \delta < b$ and $x_0 - \delta > a$, $0 < y_0 - y < \delta/2$ for all $\delta > 0$ which satisfies $y_0 + \delta < d$ and $y_0 - \delta > c$ whenever $0 < \delta < \delta_0$. Let us divide D into the sets $D_{ij} = \langle c_i, c_{i+1} \rangle \times \langle d_j, d_{j+1} \rangle$ for $1 \leq i \leq 3$ and $1 \leq j \leq 3$, where $c_1 = a$, $c_2 = x_0 - \delta$, $c_3 = x_0 + \delta$, $c_4 = b$ and $d_1 = c$, $d_2 = y_0 - \delta$, $d_3 = y_0 + \delta$, $d_4 = d$. Recall the definition of the function $g(t, s)$ such that

$$g(t, s) = \begin{cases} f(t, s), & (t, s) \in D, \\ 0, & (t, s) \in \mathbb{R}^2 \setminus D. \end{cases}$$

Let $(x_0, y_0) \in D$ is a μ -generalized Lebesgue point of function $f \in L_1(D)$. Therefore, for all given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all h, k satisfying $0 < h, k \leq \delta$, the inequality:

$$(4.3) \quad \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} |f(t, s) - f(x_0, y_0)| ds dt < \varepsilon \mu_1(h) \mu_2(k)$$

holds.

Set $I_\lambda(x, y) := |L_\lambda(f; x, y) - f(x_0, y_0)|$. According to condition (c) of class A, we shall write

$$\begin{aligned} I_\lambda(x, y) &= \left| \iint_D f(t, s) K_\lambda(t - x, s - y) ds dt - f(x_0, y_0) \right| \\ &\leq \iint_{\mathbb{R}^2} |g(t, s) - f(x_0, y_0)| |K_\lambda(t - x, s - y)| ds dt \\ &\quad + |f(x_0, y_0)| \left| \iint_{\mathbb{R}^2} K_\lambda(t - x, s - y) ds dt - 1 \right| \\ &= \iint_D |f(t, s) - f(x_0, y_0)| |K_\lambda(t - x, s - y)| ds dt \\ &\quad + |f(x_0, y_0)| \left| \iint_{\mathbb{R}^2} K_\lambda(t - x, s - y) ds dt - 1 \right| \\ &\quad + |f(x_0, y_0)| \iint_{\mathbb{R}^2 \setminus D} |K_\lambda(t - x, s - y)| ds dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

It is easy to see that $I_2 \rightarrow 0$ as $\lambda \rightarrow \lambda_0$ by condition (c) of class A. On the other hand, since

$$\begin{aligned} I_3 &\leq |f(x_0, y_0)| \iint_{\mathbb{R}^2 \setminus \langle x-\delta, x+\delta \rangle \times \langle x-\delta, x+\delta \rangle} |K_\lambda(t - x, s - y)| ds dt \\ &= |f(x_0, y_0)| \iint_{\mathbb{R}^2 \setminus \langle -\delta, +\delta \rangle \times \langle -\delta, +\delta \rangle} |K_\lambda(t, s)| ds dt, \end{aligned}$$

by condition (e) of class A we have $I_3 \rightarrow 0$ as (x, y, λ) tends to (x_0, y_0, λ_0) .

Obviously, the integral I_1 can be written as follows:

$$\begin{aligned} I_1 &= \left\{ \iint_{D \setminus D_{22}} + \iint_{D_{22}} \right\} |f(t, s) - f(x_0, y_0)| |K_\lambda(t - x, s - y)| ds dt \\ &= I_{11} + I_{12}. \end{aligned}$$

We will show that $I_{11} \rightarrow 0$ as (x, y, λ) tends to (x_0, y_0, λ_0) by following the similar steps as in [11].

Now, recall that the rectangle D was splitted into nine rectangles at the beginning of the proof. Therefore, I_{11} consists of eight integrals. Now, denote the integral which corresponds to D_{12} by $I_{D_{12}}$. Hence, by condition (f) of class A , the following inequality

$$\begin{aligned} I_{D_{12}} &= \int_a^{x_0-\delta y_0+\delta} \int_{y_0-\delta}^{y_0} |f(t, s) - f(x_0, y_0)| |K_\lambda(t-x, s-y)| ds dt \\ &\leq |K_\lambda(\delta/2, 0)| \left(\|f\|_{L_1(D)} + |f(x_0, y_0)| |b-a| |d-c| \right) \end{aligned}$$

holds. By condition (d) of class A , $I_{D_{12}} \rightarrow 0$ as (x, y, λ) tends to (x_0, y_0, λ_0) . The remaining seven integrals are evaluated by an analogous method and this part is omitted.

It follows that

$$I_{11} \leq \sup_{(t,s) \in D \setminus D_{22}} |K_\lambda(t-x, s-y)| \left(\|f\|_{L_1(D)} + |f(x_0, y_0)| |b-a| |d-c| \right).$$

Consequently, $I_{11} \rightarrow 0$ as (x, y, λ) tends to (x_0, y_0, λ_0) .

Next, we prove that I_{12} tends to zero as (x, y, λ) tends to (x_0, y_0, λ_0) . Therefore, it is easy to see that the following inequality holds for I_{12} , that is

$$\begin{aligned} I_{12} &= \left\{ \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} + \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} \right\} |f(t, s) - f(x_0, y_0)| |K_\lambda(t-x, s-y)| ds dt \\ &\quad + \left\{ \int_{x_0-\delta}^{x_0} \int_{y_0}^{y_0+\delta} + \int_{x_0}^{x_0+\delta} \int_{y_0}^{y_0+\delta} \right\} |f(t, s) - f(x_0, y_0)| |K_\lambda(t-x, s-y)| ds dt \\ &= I_{121} + I_{122} + I_{123} + I_{124}. \end{aligned}$$

Let us consider the integral I_{121} . For the evaluations, we need to define the following variations:

$$\begin{aligned} A_1(u, v) &: = \begin{cases} \bigvee_u^{x_0+\delta-x} \bigvee_{y_0-\delta-y}^v |K_\lambda(t, s)|, & x_0 - x \leq u < x_0 + \delta - x, \\ & y_0 - \delta - y < v \leq y_0 - y, \\ 0, & \text{otherwise.} \end{cases} \\ A_2(u) &: = \begin{cases} \bigvee_u^{x_0+\delta-x} |K_\lambda(t, y_0 - \delta - y)|, & x_0 - x \leq u < x_0 + \delta - x, \\ 0, & \text{otherwise.} \end{cases} \\ A_3(v) &: = \begin{cases} \bigvee_{y_0-\delta-y}^v |K_\lambda(x_0 - x + \delta, y)|, & y_0 - \delta - y < v \leq y_0 - y, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Taking above variations and (4.3) into account and applying bivariate integration by parts method (see, e.g., [8]) to last inequality, we have

$$\begin{aligned} I_{121} &\leq -\varepsilon \int_{x_0-x}^{x_0-x+\delta} \int_{y_0-y-\delta}^{y_0-y} [A_1(t, s) + A_2(t) + A_3(s) + |K_\lambda(x_0 - x + \delta, y_0 - \delta - y)|] \\ &\quad \times \{\mu_1(t - x_0 + x)\}'_t \{\mu_2(y_0 - s - y)\}'_s ds dt \\ &= \varepsilon (i_1 + i_2 + i_3 + i_4). \end{aligned}$$

For the similar situation, we refer the reader to [8, 10].

Now, using Remark 1 and condition (f) of class A , we get

$$\begin{aligned} i_1 + i_2 + i_3 + i_4 &= - \int_{x_0-x}^{x_0-x+\delta} \int_{y_0-y-\delta}^0 |K_\lambda(t, s)| \{\mu_1(t - x_0 + x)\}'_t \{\mu_2(y_0 - s - y)\}'_s ds dt \\ &\quad + \int_{x_0-x}^{x_0-x+\delta} \int_0^{y_0-y} (|K_\lambda(t, s)| - 2K_\lambda(t, 0)) \{\mu_1(t - x_0 + x)\}'_t \{\mu_2(y_0 - s - y)\}'_s ds dt. \end{aligned}$$

Hence, the following inequality holds for I_{121} :

$$\begin{aligned} I_{121} &\leq \varepsilon \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} |K_\lambda(t - x, s - y)| |\{\mu_1(t - x_0)\}'_t| |\{\mu_2(y_0 - s)\}'_s| ds dt \\ &\quad + 2\varepsilon |K_\lambda(0, 0)| \mu_1(\delta) \mu_2(|y_0 - y|). \end{aligned}$$

Analogous computations for I_{122} , I_{123} and I_{124} yield:

$$\begin{aligned} I_{122} &\leq \varepsilon \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} |K_\lambda(t - x, s - y)| |\{\mu_1(x_0 - t)\}'_t| |\{\mu_2(y_0 - s)\}'_s| ds dt \\ &\quad + 2\varepsilon |K_\lambda(0, 0)| (\mu_1(\delta) \mu_2(|y_0 - y|) + \mu_2(\delta) \mu_1(|x_0 - x|)) + 4\varepsilon K_\lambda(0) \mu_1(|x_0 - x|) \mu_2(|y_0 - y|), \\ I_{123} &\leq \varepsilon \int_{x_0-\delta}^{x_0} \int_{y_0}^{y_0+\delta} |K_\lambda(t - x, s - y)| |\{\mu_1(x_0 - t)\}'_t| |\{\mu_2(s - y_0)\}'_s| ds dt \\ &\quad + 2\varepsilon |K_\lambda(0, 0)| \mu_2(\delta) \mu_1(|x_0 - x|), \\ I_{124} &\leq \varepsilon \int_{x_0}^{x_0+\delta} \int_{y_0}^{y_0+\delta} |K_\lambda(t - x, s - y)| |\{\mu_1(t - x_0)\}'_t| |\{\mu_2(s - y_0)\}'_s| ds dt. \end{aligned}$$

Hence the following inequality is obtained for I_{12} :

$$\begin{aligned} I_{12} &\leq \varepsilon \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} |K_\lambda(t - x, s - y)| |\{\mu_1(|x_0 - t|)\}'_t| |\{\mu_2(|y_0 - s|)\}'_s| ds dt \\ &\quad + 4\varepsilon |K_\lambda(0, 0)| (\mu_1(\delta) \mu_2(|y_0 - y|) + \mu_2(\delta) \mu_1(|x_0 - x|)) \\ &\quad + 4\varepsilon |K_\lambda(0, 0)| \mu_1(|x_0 - x|) \mu_2(|y_0 - y|) \end{aligned}$$

or equivalently,

$$\begin{aligned}
I_{12} &\leq \varepsilon \int_{x_0-\delta y_0-\delta}^{x_0+\delta y_0+\delta} \int K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) \rho_1(|t-x_0|) \rho_2(|s-y_0|) ds dt \\
&\quad + 4\varepsilon |K_\lambda(0,0)| (\mu_1(\delta) \mu_2(|y_0-y|) + \mu_2(\delta) \mu_1(|x_0-x|)) \\
&\quad + 4\varepsilon |K_\lambda(0,0)| \mu_1(|x_0-x|) \mu_2(|y_0-y|).
\end{aligned}$$

The remaining part of the proof is obvious by the hypotheses (4.1) and (4.2). Thus the proof is completed. \square

The following theorem gives a pointwise approximation of the integral operators of type (1.3) to the function f at μ -generalized Lebesgue point of $f \in L_1(\mathbb{R}^2)$.

Theorem 2. *Suppose that the hypotheses of Theorem 1 are satisfied for $D = \mathbb{R}^2$. If (x_0, y_0) is a μ -generalized Lebesgue point of $f \in L_1(\mathbb{R}^2)$, then*

$$\lim_{(x,y,\lambda) \rightarrow (x_0,y_0,\lambda_0)} L_\lambda(f; x, y) = f(x_0, y_0).$$

Proof. The proof of this theorem is quite similar to proof of preceding one and thus, it is omitted. \square

5. Rate of Convergence

In this section, we give a theorem concerning the rate of pointwise convergence of the operators of type (1.3).

Theorem 3. *Suppose that the hypotheses of Theorem 1 (or Theorem 2) are satisfied.*

Let

$$\Delta(\lambda, \delta, x, y) = \int_{x_0-\delta y_0-\delta}^{x_0+\delta y_0+\delta} \int |K_\lambda(t-x, s-y)| \rho_1(|t-x_0|) \rho_2(|s-y_0|) ds dt$$

for $0 < \delta < \delta_0$, and the following assumptions are satisfied:

i: $\Delta(\lambda, \delta, x, y) \rightarrow 0$ as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$ for some $\delta > 0$.

ii: For every $\gamma > 0$,

$$|K_\lambda(\gamma, 0)| = |K_\lambda(0, \gamma)| = o(\Delta(\lambda, \delta, x, y))$$

as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$.

iii: For every $\gamma > 0$,

$$\lim_{\lambda \rightarrow \lambda_0} \iint_{\mathbb{R}^2 \setminus \langle -\gamma, \gamma \rangle \times \langle -\gamma, \gamma \rangle} |K_\lambda(t, s)| ds dt = o(\Delta(\lambda, \delta, x, y))$$

as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$.

iv: As $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$, we have

$$\left| \iint_{\mathbb{R}^2} K_\lambda(t-x, s-y) ds dt - 1 \right| = o(\Delta(\lambda, \delta, x, y)).$$

Then, at each μ -generalized Lebesgue point of $f \in L_1(D)$ we have

$$|L_\lambda(f; x, y) - f(x_0, y_0)| = o(\Delta(\lambda, \delta, x, y))$$

as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$.

Proof. By the hypotheses of Theorem 1, we can write for $\delta > 0$

$$\begin{aligned} |L_\lambda(f; x, y) - f(x_0, y_0)| &\leq |f(x_0, y_0)| \left| \iint_{\mathbb{R}^2} K_\lambda(t - x, s - y) ds dt - 1 \right| \\ &\quad + |f(x_0, y_0)| \iint_{\mathbb{R}^2 \setminus \langle -\delta, +\delta \rangle \times \langle -\delta, +\delta \rangle} |K_\lambda(t, s)| ds dt \\ &\quad + \sup_{(t, s) \in D \setminus D_{22}} |K_\lambda(t - x, s - y)| \left(\|f\|_{L_1(D)} + |f(x_0, y_0)| |b - a| |d - c| \right) \\ &\quad + \varepsilon \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} K_\lambda \left(\sqrt{(t - x)^2 + (s - y)^2} \right) \rho_1(|t - x_0|) \rho_2(|s - y_0|) ds dt \\ &\quad + 4\varepsilon |K_\lambda(0, 0)| (\mu_1(\delta) \mu_2(|y_0 - y|) + \mu_2(\delta) \mu_1(|x_0 - x|)) \\ &\quad + 4\varepsilon |K_\lambda(0, 0)| \mu_1(|x_0 - x|) \mu_2(|y_0 - y|). \end{aligned}$$

From (i)-(iv) and using class A conditions, we have the desired result for the case D is bounded, that is

$$|L_\lambda(f; x, y) - f(x_0, y_0)| = o(\Delta(\lambda, \delta, x, y)).$$

One may prove the assertion for the case $D = \mathbb{R}^2$ following the same steps. Thus, the proof is completed. \square

Example 1. In this example, we used two dimensional counterpart of the kernel function used in [6].

Let $\Lambda = [1, \infty)$, $\lambda_0 = \infty$ and

$$K_\lambda(t, s) = \begin{cases} \lambda^2, & (t, s) \in [0, 1/\lambda] \times [0, 1/\lambda], \\ 0, & \mathbb{R}^2 \setminus [0, 1/\lambda] \times [0, 1/\lambda]. \end{cases}$$

It is easy to check that given $K_\lambda(t, s)$ is from the class A . Let $\mu_1(t) = t$, $\mu_2(s) = s$. Hence, we obtain

$$\begin{aligned} \Delta(\lambda, \delta, x, y) &= \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} |K_\lambda(t - x, s - y)| \left| \{\mu_1(|t - x_0|)\}'_t \right| \left| \{\mu_2(|s - y_0|)\}'_s \right| ds dt \\ &= 4\delta^2 \lambda^2. \end{aligned}$$

In order to find for which $\delta > 0$ the condition (i) in Theorem 1 is satisfied, let $\Delta(\lambda, \delta, x, y) \rightarrow 0$ as $(x, y, \lambda) \rightarrow (x_0, y_0, \infty)$. Hence

$$\lim_{(x, y, \lambda) \rightarrow (0, 0, 0)} \Delta(\lambda, \delta, x, y) = 0$$

if and only if $\delta = o(1/\lambda)$. Consequently, the following equation

$$\Delta(\lambda, \delta, x, y) = \iint_{[0, 1/\lambda^{1+\alpha}] \times [0, 1/\lambda^{1+\alpha}]} \lambda^2 ds dt$$

holds for $\alpha \in (0, \infty)$. From above equation we see that

$$\Delta(\lambda, \delta, x, y) = O(1/\lambda^{1+\alpha}).$$

By definition of $K_\lambda(t, s)$, the conditions (ii) and (iii) of Theorem 5.1 are satisfied.

The last terms which must be considered are $K_\lambda(0, 0) \mu_2(|y - y_0|)$ and $K_\lambda(0, 0) \mu_1(|x - x_0|)$. Analyzing the following limits, we see that

$$\begin{aligned} \lim_{(x, y, \lambda) \rightarrow (0, 0, 0)} K_\lambda(0, 0) \mu_1 |x - x_0| &= \lambda^2 |x - x_0| = c < \infty, \\ \lim_{(x, y, \lambda) \rightarrow (0, 0, 0)} K_\lambda(0, 0) \mu_2 |y - y_0| &= \lambda^2 |y - y_0| = c' < \infty, \end{aligned}$$

if and only if the rates of convergence of $\lambda^2 \rightarrow \infty$ and $x \rightarrow x_0$, $\lambda^2 \rightarrow \infty$ and $y \rightarrow y_0$ are equivalent.

Hence

$$|L_\lambda(f; x, y) - f(x_0, y_0)| = o(\Delta(\lambda, \delta, x, y)) = o(1/\lambda^{1+\alpha}).$$

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ON THE APPROXIMATION BY CONVOLUTION TYPE DOUBLE SINGULAR INTEGRAL OPERATORS

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